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Equilibrium shapes of twisted magnetic filaments

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Abstract

It is shown that ferromagnetic filaments with free and unclamped ends undergo buckling instabilities under the action of twist. Solutions of nonlinear equations describing the buckled shapes are found, and it is shown that the transition to the buckled shape is subcritical if the magnetization is parallel to the field and supercritical when the magnetization of the straight filament is opposite to the external field. Solutions with the localized curvature distribution are found in the case of long filaments. The class of solutions corresponding to helices is described, and the behavior of coiled ferromagnetic and superparamagnetic filaments is compared.

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1. Introduction

Flexible ferromagnetic filaments are used by magnetotactic bacteria for navigation in the magnetic field of the Earth [1]. They can be made artificially by linking commercially available ferromagnetic particles functionalized by streptavidin with biotinized DNA fragments [2]. The investigation of their behavior in an ac magnetic field and at the magnetic field inversion has shown that even short filaments possess quite large values of the magnetoelastic number Cm that expresses the ratio of magnetic and elastic torques. Thus, these filaments may be used for the creation of magnetically driven microengines of new types [3]. For the analysis of magnetically driven microengines, the knowledge of their behavior under different loadings is required. The necessary spatial symmetry breaking may be caused by the buckling instabilities [4, 5]. For example, it has recently been shown that ferromagnetic filaments possess chiral symmetry breaking oscillatory instability under the action of the twist and the magnetic field which leads to their rotation [6].

The buckling of elastic rods in a magnetic field has been studied previously in different situations. The buckling of an extensible string due to the Lorentz force on the current in the external field is considered in [7, 8]. The interest in this problem is caused by the creation of tethered satellite systems [9].

It is interesting to remark here that, in spite of completely different physics, the formation of helicoidal shapes takes place also in the case of ferromagnetic strings in the magnetic field. This may be illustrated as follows. The stress $\vec{F}$ arising in an extensible ferromagnetic string reads $\vec{F} = T(\varepsilon)\vec{e}_3 - MH\,\mathrm{Re}(\vec{e}^*\vec{h}\cdot\vec{e})$ [10] ($\vec{e}_3$ is the tangent vector, ε is the extension of the filament, $T(\varepsilon)$ is the tension, M is the magnetization of filament per unit length, $\vec{h}$ is the unit vector along the external field, H is the field strength and $\vec{e} = \vec{n} + i\vec{b}$ ($i^2 = -1$), $\vec{n}, \vec{b}$ are normal and binormal vectors, respectively). The equilibrium condition $\mathrm{d}\vec{F}/\mathrm{d}l = 0$ for the ferromagnetic string reads $k(T + MH\vec{e}_3\cdot h) = 0$, $\mathrm{d}T/\mathrm{d}l + MHk\vec{h}\cdot\vec{n} = 0$, where k is its curvature. These equations have a non-trivial solution ($k \neq 0$) that corresponds to a helix with its axis along the field. For the helix, we have $\vec{h}\cdot\vec{n} = 0$ hence $T = \mathrm{const} > 0$ and $\vec{e}_3\cdot\vec{h} < 0$. For the string given in the parametric form $\vec{r} = \vec{r}(l_0), l_0 \in [0, L_0]$ with $\vec{r}(0) = 0; \vec{r}(L_0) = c\vec{e}_z$ (L_0 is the string length in the unstretched state, l_0 is the arclength of material points in the unstretched state, $c > L_0$) the extension $\varepsilon = c/(L_0\cos\vartheta) - 1$, where ϑ is the angle between the tangent $\vec{e}_3$ and the axis of the helix parallel to $\vec{e}_z$. By the equation of state $T = S(L - L_0)/L_0$ (S is the stretching modulus), the equilibrium condition gives $S(c/(L_0\cos\vartheta) - 1) = MH\cos\vartheta$. This, for the critical value of the field strength applied opposite to the magnetization of the string, gives $(MH)_c = S(c/L_0 - 1)$. If the field strength is large, the angle of the helix is given by $\cos\vartheta = \sqrt{Sc/(L_0MH)}$.

The physics of the helix formation by the ferromagnetic string is rather simple: if the magnetic field is applied opposite to the magnetization of the stretched magnetized string, it tries to overturn to the direction of the field. The string does it by local stretching and by orienting its local elements at some angle with respect to the field. Since the tension in the string is constant, the angle of the tangent is also constant. That is possible only in the case of the helix.

For real ferromagnetic filaments, the bending and twisting modulus should be taken into account. The theoretical analysis of the buckling instabilities of ferromagnetic filaments under the inversion of the magnetic field is carried out in [4]. Experimental observations of the buckling instability of the ferromagnetic filaments synthesized by linking micron-sized ferromagnetic particles by 1000 bp long DNA fragments are carried out in [2]. From the comparison of the experimentally observed shape dynamics of the filaments with that obtained numerically from the same initial conditions, it follows that the value of magnetoelastic number Cm for rather short filaments, used in the experiments, is approximately equal to 10.

If an ac magnetic field is applied, different phenomena take place in dependence on its frequency. If the frequency is high enough, the ferromagnetic filament orients perpendicularly to the field [2, 4]. The value of the magnetoelastic number, determined from the time dependence of the filament orientation angle, is close to that determined from the dynamics of the filament at the magnetic field inversion [2].

It is suggested in [11] that one can obtain information about the cytoskeleton of magnetotactic bacteria using the effect of orientation of ferromagnetic filaments perpendicularly to the ac magnetic field. In the case when the length of the chain of magnetosomes is much less than the length L of the semiflexible polymer to which it is attached, it is possible to consider the deformation of the semiflexible polymer induced by the torque on the magnetic dipole situated in its center. Then for the sum of mean per period magnetic and elastic energies of the attached filament, we have $E = -mH^2\sin^2\beta/4H_c$, where β is the angle which the filament makes with the field direction, m is the total magnetic moment of the chain of magnetosomes. Coercitive force $H_c = 6l_pH_L/L$ (l_p is the persistence length of the biopolymer, $H_L = k_BT/m$) characterizes the coupling of the magnetic filament to the elastic network. This energy is similar to the mean energy of the chain of the ferromagnetic particles interacting by magnetodipolar forces in an ac field, when $H_c = 1.2\pi M_s$ (M_s is the saturation magnetization of the ferromagnetic particles). These estimates show that if the persistence length of the bacterium cytoskeleton corresponds to the persistence length of actin, then H_c due to the flexibility of the network is much less than the coercive force due to the magnetodipolar interactions. Although existing experimental data show that magnetotactic bacteria align perpendicularly to the ac magnetic field nevertheless they do not give the evidence that this is due to the flexibility of cytoskeleton to which the chain of magnetosomes is attached [11].

At present moment, the mechanical properties of the ferromagnetic filaments of magnetotactic bacteria are not investigated sufficiently. There is only estimate of the bending modulus A of the filaments due to the magnetodipolar interactions: $A = m_p^2/2d^2$ (m_p is the magnetic moment of the magnetosomes, d is their diameter) [12], which allow one to explain the formation of the rings of chains of the ferromagnetic particles [12] and estimate the stress necessary for their buckling [13, 14]. The estimate of the bending modulus shows that the magnetoelastic number MHL^2/A of the chain of ferromagnetic particles may be calculated as $3HN^2/\pi M_s$, where N is the number of particles in the chain. Thus quite real are the values of the magnetoelastic number about 10^3 or more. Numerical simulation data on the behavior of the chains of superparamagnetic particles in the shear flow containing up to 10^2 particles are given in [15]. The data correspond quite well to the estimates obtained by using the value of bending modulus due to the magnetodipolar interactions close to given above.

Here the equilibrium shapes due to the buckling of magnetic filaments under the action of the twist and magnetic fields are analyzed. Some preliminary results on the linear stability are given in [2, 6]. The buckling of elastic rods under the action of twist is the well-known problem of solid state mechanics (see, for example, [16–18]).

The model and the basic equations are introduced in section 2, the localized solution for the curvature is obtained in section 3, the helices are described in section 4 and the buckled shapes of filaments with a finite length are considered in section 5.

2. Model

The model of the twisted ferromagnetic filament is based on the Kirchhoff model of the elastic rod extended to account for interaction energy of the filament with an external field [10] (l is the contour length of the filament, subscript $_{,...}$ denotes derivative). Its energy reads

$$E = \frac{1}{2}A\int\left(\Omega_1^2+\Omega_2^2\right)\mathrm{d}l + \frac{1}{2}C\int\Omega_3^2\,\mathrm{d}l - MH\int\vec{e}_3\cdot\vec{h}\,\mathrm{d}l - \int\Lambda\,\mathrm{d}l. \qquad (1)$$

Here M is the magnetization of the filament per its unit length, which is oriented along the tangent vector $\vec{e}_3$ of the filament. The unit vectors $\vec{e}_i$ of the material frame obey $\vec{e}_{i,l} = \vec{\Omega}\times\vec{e}_i (i = 1, 2, 3,$ the vector $\vec{\Omega}$ characterizes the bending and twisting strain of the filament). A and C are the bend and twist elastic constants, H is the applied magnetic field strength and $\vec{h}$ is the unit vector along the applied magnetic field. Inextensibility of the filament is enforced by the term $-\int\Lambda \mathrm{d}l$ in (1). The Lagrange multiplier Λ describes the tension enforcing the inextensibility. By the use of the complex curvature $\psi = (-\mathrm{i}\Omega_1+\Omega_2)\exp\left(\mathrm{i}\int^l\Omega_3\mathrm{d}l'\right)$ and $\vec{\epsilon} = (\vec{e}_1+i\vec{e}_2)\exp\left(\mathrm{d}\int^l\Omega_3\,\mathrm{d}l'\right)$ and by the standard variational procedure [10, 19], the following expressions for the stress $\vec{F}$ and the couple stress $\vec{T}$ of the ferromagnetic filament under the action of the magnetic field are derived [2, 6]:

$$\vec{F} = \mathrm{Re}[\vec{\epsilon}^*(-A\psi_{,l} + \mathrm{i}C\Omega_3\psi - MH\vec{h}\cdot\vec{\epsilon})] - \vec{e}_3\left(\tfrac{1}{2}A|\psi|^2+\Lambda\right), \qquad (2)$$

$$\vec{T} = -A\mathrm{Im}(\psi\vec{\epsilon}^*) + C\Omega_3\vec{e}_3. \qquad (3)$$

In the case of superparamagnetic filament [5, 10], the magnetic field-dependent term $\vec{F}_m = -\mathrm{Re}[\vec{\epsilon}^* M H \vec{h} \cdot \epsilon]$ in the relation for the stress (2) is given as follows

$$\vec{F}_m = \mathrm{Re}[\vec{\epsilon}^*(-\Delta M H \vec{e}_3 \cdot \vec{h}\vec{h} \cdot \vec{\epsilon})], \tag{4}$$

where

$$\Delta M = \pi a^2 \frac{(\mu - 1)^2 H}{4\pi(\mu + 1)}$$

is the part of superparamagnetic filament magnetization that depends on its orientation due to the demagnetizing field effect (a is the radius of the circular cross section of the filament and μ is its magnetic permeability).

It would be relevant to notify here the difference between the magnetization mechanisms of the ferromagnetic and superparamagnetic filaments. The ferromagnetic filaments have spontaneous magnetization aligned along the tangent vector of the filament. As an example, we have the chain of the ferromagnetic nanoparticles in magnetotactic bacteria. Magnetic moments of particles are parallel due to the strong dipolar coupling. The magnetization of the superparamagnetic filaments arises only in an external field and due to the long-range dipolar interactions depends on the orientation of the magnetic field with respect to the axis of filament. The dependence of the magnetization on the direction of the applied magnetic field gives the torque on the filament and the normal force described by the relation (4). It is also necessary to emphasize that the effects of the magnetic field on the filaments considered in this paper are determined by the torques. Effects of the Kelvin force $M\nabla H$ in the non-uniform applied fields are worthy of separate investigation.

In the following, the equilibrium shapes of the filament with free and unclamped ends are considered. The equation of the torque balance

$$\vec{T}_{,l} + \vec{e}_3 \times \vec{F} + M H \vec{e}_3 \times \vec{h} = 0 \tag{5}$$

implies $C\Omega_3 =$ const. Since for a free rod in the equilibrium we have $\vec{F} = 0$, relation (2) gives

$$-A\psi_{,l} + \mathrm{i}C\Omega_3\psi - M H \vec{h} \cdot \vec{\epsilon} = 0 \tag{6}$$

and

$$\tfrac{1}{2}A|\psi|^2 + \Lambda = 0. \tag{7}$$

Vectors $\vec{e}_3, \vec{\epsilon}$ obey the equations

$$\frac{\mathrm{d}\vec{e}_3}{\mathrm{d}l} = \mathrm{Re}(\psi\vec{\epsilon}^*), \tag{8}$$

$$\frac{\mathrm{d}\vec{\epsilon}}{\mathrm{d}l} = -\psi\vec{e}_3, \tag{9}$$

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}l} = \vec{e}_3. \tag{10}$$

The shape of the filament can be calculated using equations (8)–(10), if the complex curvature and the boundary conditions for vectors $\vec{e}_3, \vec{\epsilon}, \vec{r}$ are known.

Equation (6) is not intrinsic since it contains the term dependent on the orientation of the filament. Using equations (8)–(9), it is possible to obtain the equation containing only the intrinsic variables of the center line of the filament. The equilibrium condition $\vec{F}_{,l} = 0$ gives

$$-A\psi_{,ll} + \mathrm{i}C\Omega_3\psi_{,l} + \left(M H \vec{e}_3 \cdot \vec{h} - \Lambda - \tfrac{1}{2}A|\psi|^2\right)\psi = 0 \tag{11}$$

and

$$-\Lambda_{,l} + MH\,\mathrm{Re}(\psi\vec{h}\cdot\vec{\epsilon}^{*}) = 0. \qquad (12)$$

Let us define $W = -\Lambda + MH\vec{e}_3\cdot\vec{h}$. Then equation (12) can be written as

$$W_{,l} = 0. \qquad (13)$$

As a result, equation (11) reads

$$-A\psi_{,ll} + \mathrm{i}C\Omega_3\psi_{,l} - \tfrac{1}{2}A|\psi|^2\psi + W\psi = 0, \qquad (14)$$

where W is constant because of (13). It can be found from the boundary conditions. Since for the filament with free and unclamped ends $\psi(\pm L) = 0$ and $\Lambda(\pm L) = 0$ we have $W = MH\vec{e}_3(\pm L)\cdot\vec{h}$, relation (7) gives

$$MH\,\vec{e}_3\cdot\vec{h} = W - \tfrac{1}{2}A|\psi|^2 \qquad (15)$$

that will be useful for the discussion of the steady shapes of the filament.

3. Localized solution

In the case of a long twisted filament, it is possible to find the localized solution of equation (14) such that $\psi(\pm\infty) = 0$ and $\vec{e}_3(\pm\infty) = \vec{h}$. In this case, the characteristic distance L_* is determined by the twist: $L_* = A/C\Omega_3(\Omega_3 > 0)$. By scaling $\psi = \tilde{\psi}/L_*$ and $l = \tilde{l}L_*$ (tildas further are omitted) and by putting $\psi = \exp(\mathrm{i}l/2)f(l)$ we obtain the following equation for the function f:

$$f_{,ll} + \tfrac{1}{2}|f|^2 f = \left(Tm - \tfrac{1}{4}\right)f. \qquad (16)$$

We see that the shape of the filament is determined by the magnetotwisting number $Tm = MHA/(C\Omega_3)^2$ that characterizes the ratio of the magnetic torque MHL_* to the twisting torque $C\Omega_3$. The real solution of equation (16) satisfying the boundary conditions reads

$$f = \frac{\sqrt{4Tm-1}}{\cosh(\sqrt{4Tm-1}\,l/2)}. \qquad (17)$$

It exists for values of the magnetotwisting number $Tm > 1/4$.

The tangent angle can be found from the integral (15), which reads

$$\tfrac{1}{2}f^2 = Tm(1 - \vec{e}_3\cdot\vec{h}). \qquad (18)$$

Relation (18) shows that for angle $\vartheta(0)$, between the tangent and the magnetic field at $l = 0$, the following relation $\sin^2(\vartheta(0)/2) = 1 - (4Tm)^{-1}$ is valid.

The shape of the filament corresponding to the localized curvature distribution can be found by integration of equations (8)–(10) at the boundary conditions $\vec{r}(-\infty) = 0$; $\vec{e}_3(-\infty) = (0, 0, 1)$; $\vec{e}_1(-\infty) = (1, 0, 0)$; $\vec{e}_2(-\infty) = (0, 1, 0)$. The shape obtained by the integration in the interval $l = [-200; 200]$ at $Tm = 0.2506$ is shown in figure 1.

As we can see, the shape of the filament depends on the ratio of the characteristic magnetic healing length $L_H = \sqrt{A/MH}$ and L_*. Near the threshold ($L_H \simeq 2L_*$), the spatial extension of the buckled region due to the stabilizing action of the magnetic field is large, and we observe the shape shown in figure 1. For the subcritical values of the twist, the loop with a radius approximately equal to L_H is formed. For example, the characteristic shape of the filament for $Tm = 1/2$ and $\vartheta(0) = \pi/2$ is shown in figure 2. We see the formation of the loop perpendicular to the field on one side of the filament.

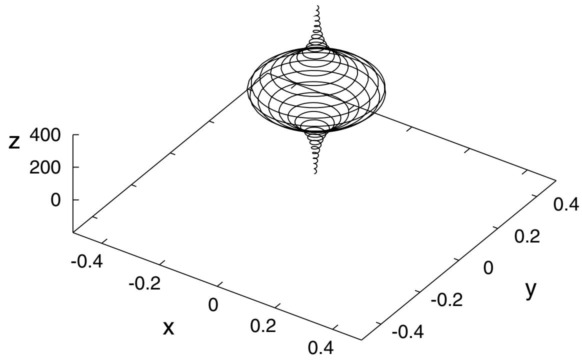

Figure 1. Localized shape perturbation of twisted ferromagnetic filament. $Tm = 0.2506$.

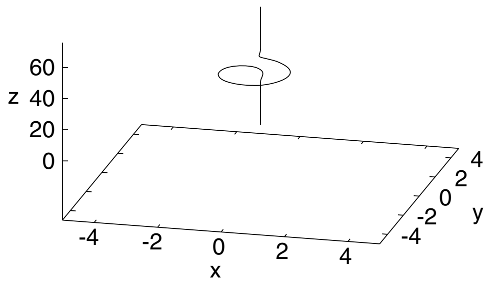

Figure 2. Localized shape perturbation of twisted ferromagnetic filament. $Tm = 0.5$.

The distance between the ends of the long filament with a curvature given by the localized solution (17) is less than the length of the straight filament by

$$D = \int_{-\infty}^{\infty} (1 - \cos\vartheta)\,\mathrm{d}l.$$

By equations (17) and (18) for the length D adsorbed due to bending, we find

$$D = \frac{A}{C\Omega_3}\frac{2\sqrt{4Tm-1}}{Tm}. \tag{19}$$

The length D attains its maximal value $4A/C\Omega_3$ at $Tm = 1/2$ that corresponds to the critical torque $C\Omega_{3c} = \sqrt{2MHA}$.

The square root dependence of the radius of the loop on the field may be obtained by a simple analysis of the energy of the filament [20]. When a loop in the form of a circle with radius r is formed, the magnetic energy increases by $MH2\pi r$ and the bending energy by $A\pi/r$. The increase of the total energy $2\pi(MHr + A/2r)$ comes from the work of the twisting torque on one turn: $2\pi C\Omega_3$. The minimum of the energy is attained for $r = L_H/\sqrt{2}$ that gives the critical torque $\sqrt{2MHA}$ corresponding to $Tm = 1/2$. As mentioned in [21], this value of the torque corresponding to $Tm = 1/2$ is critical for the formation of a plectoneme, when the loop flips to the plane of the magnetic field.

Interesting conclusions may be obtained from the analysis of the energy of a bent twisted filament. In the case of a constant twisting torque T_3 its work $-T_3R$ should be taken into account [22], where R is the rotation angle of the filament. The relation $R = 2\pi Lk$ allows us to express the angle R using the linking number $Lk = Tw + Wr$ [23] that is the sum of the twist Tw and writhe Wr. If the twist Tw is large then it is energetically advantageous to increase the angle R by the writhing of the filament. According to Fuller [24], the local density of the writhe wr may be expressed as

$$2\pi wr = \frac{\vec{e}_z \cdot \vec{e}_3 \times \mathrm{d}\vec{e}_3/\mathrm{d}l}{1 + \cos\vartheta}. \tag{20}$$

Using equation (5), equation (8) and $T_z = C\Omega_3$ [16], the last expression can be rewritten as

$$2\pi wr = \frac{C\Omega_3}{A}\frac{1 - \cos\vartheta}{1 + \cos\vartheta}. \tag{21}$$

By equations (17) and (18), and the integration for the filament with length $2L \gg L_*$, we have

$$2\pi Wr = 4\arctan\sqrt{4Tm - 1}.$$

By writhing, the rotation angle R is increased without further twisting. It means that the energy of the twist is diminished in comparison with the straight filament by amount $-C\Omega_3 4\arctan\sqrt{4Tm - 1}$. The writhing makes the filament bend that leads to the increase of the bending and magnetic energies. By relation (1) and (18), we have $C\Omega_3\int_{-\infty}^{\infty} f^2\,\mathrm{d}l - MH2L$ for this increase. As a result, the positive difference of energies of the long writhed filament and the straight rod is [25]

$$\Delta E = C\Omega_3(4\sqrt{4Tm - 1} - 4\arctan\sqrt{4Tm - 1}). \tag{22}$$

The same result follows from the consideration of the filament at a constant rotation angle R. In this case, the twist diminishes at writhing by $\Delta\Omega_3 2L = -4\arctan\sqrt{4Tm - 1}$. The corresponding decrease of the twisting energy for a long filament, when $\Delta\Omega_3$ is small, is

$$2LC\Omega_3\Delta\Omega_3 = -C\Omega_3 4\arctan\sqrt{4Tm - 1}.$$

Accounting for a change of the bending and magnetic energies, we obtain the relation (22).

At $Tm = 1/4$ that corresponds to the threshold of the buckling instability of an infinite twisted filament $\Delta E = 0$. For $Tm < 1/4$, the straight shape of the filament is unstable. The localized solution exists at $Tm > 1/4$. This is in the range of the magnetotwisting number corresponding to the stability region of the straight shape. In this case, $\Delta E > 0$ indicates that the bifurcation to the buckled shapes of the long filaments is subcritical. The conclusion about the subcritical bifurcation to the buckled shape is also in agreement with the result of [26] that shows that the loop solution of the untwisted filament is unstable with respect to the translation of the loop to one of the ends of the filament. In real situations, the loop may be stabilized by the friction at the self-contact of the filament. The exact analysis carried out in section 5 confirms the subcritical character of the bifurcation to the buckled shapes for the long filaments.

For large Tm, the relation (22) may be interpreted as follows. Equations for a ferromagnetic filament in the equilibrium have a solution in the form of the loop having the energy $8\sqrt{MHA}$ [26]. The twisted filament buckles and the arising loop flips to the plane containing the external field. Linking number increases by 1 and the work carried out by the external torque is $2\pi C\Omega_3$. The energy increase is $8\sqrt{MHA} - 2\pi C\Omega_3$—the one given by equation (22) in the limit of large Tm.

4. Helices

Localized solution is a particular solution describing the equilibrium of the long filament under the action of the twist and the magnetic field. Experimentally localized states for twisted rubber rods with controlled distance between the ends are observed in [16] at post-buckling transformation of the helice. The tension in these experiments responds passively. Ferromagnetic filaments give the possibility to carry out the buckling experiments with twisted rods at controlled tension, which role for the free filaments is played, as one can see from the energy functional (1) by the magnetic field. The twisting torque may be created, for example, by attaching magnetic beads to the ends of the filament as in the magnetic tweezer experiments [20].

General solutions may be obtained from the equations expressing the condition of zero stress in the filament (6) and (7). Multiplying equation (6) by ψ^*, its complex conjugate by ψ, adding and taking into account relation (8) we have

$$-(A|\psi|^2)_{,l} - MH2\vec{h}\cdot\frac{\mathrm{d}\vec{e}_3}{\mathrm{d}l} = 0.$$

This gives the first integral W of the equilibrium equations

$$\tfrac{1}{2}A|\psi|^2 + MH\vec{h}\cdot\vec{e}_3 = W, \tag{23}$$

where $W_{,l} = 0$. The relation (23), $|\psi|^2 = \Omega_1^2 + \Omega_2^2$, and two relations which follow from the torque balance

$$\frac{\mathrm{d}T_3}{\mathrm{d}l} = 0, \tag{24}$$

$$\frac{\mathrm{d}T_z}{\mathrm{d}l} = 0 \tag{25}$$

show the analogy between the integration of the equations of a symmetric heavy top and the elastic rod [27]. Introducing the Euler angles $\vartheta, \varphi, \alpha$ as in [28] (ϑ is the tangent angle with respect to the direction of the magnetic field), scalings as in section 3, expressing $\Omega_1^2 + \Omega_2^2 = \vartheta_{,l}^2 + \varphi_{,l}^2\sin^2\vartheta$ and using $T_z = C\Omega_3$ [16], the integral W reads

$$W = \tfrac{1}{2}\vartheta_{,l}^2 + U_{\mathrm{eff}}(\vartheta). \tag{26}$$

The effective potential

$$U_{\mathrm{eff}}(\vartheta) = \frac{1}{2}\cdot\frac{1-\cos\vartheta}{1+\cos\vartheta} + Tm\cos\vartheta$$

at $Tm > 1/4$ has minimum at ϑ given by the relation

$$\vartheta = 2\arccos\left((4Tm)^{1/4}\right). \tag{27}$$

The minimal value of the effective potential is $\min(U_{\mathrm{eff}}) = 1/2 - (1-\sqrt{Tm})^2$. If $Tm > 1/4$ and $W = \min(U_{\mathrm{eff}}) = 1/2 - (1-\sqrt{Tm})^2$ then $\vartheta = \mathrm{const}$ that corresponds to a helix. If $1/2 - (1-\sqrt{Tm})^2 < W < Tm$, the tangent angle ϑ is in the interval $[\vartheta_1, \vartheta_2]$. At the value of the integral $W = U_{\mathrm{eff}}(0) = Tm$, we obtain the localized solution. Several shapes corresponding to the intermediate values of Tm and the particular values of the integral W defined by $W = U_{\mathrm{eff}}(\vartheta_1)$ are shown in figure 3.

The analogy with the symmetric heavy top is no longer valid for the superparamagnetic filament. Due to this, it is impossible to obtain such a family of solutions as described for the ferromagnetic filaments. Nevertheless, it is possible to obtain the solution corresponding to the helical shape of the superparamagnetic filament. The condition of zero stress is

$$-A\psi_{,l} + \mathrm{i}C\Omega_3\psi - \Delta MH\vec{e}_3\cdot\vec{h}\vec{h}\cdot\vec{\epsilon} = 0, \tag{28}$$

where $\psi = k\exp(\mathrm{i}\tau l)$, $\vec{\epsilon} = \exp(\mathrm{i}\tau l)(\vec{n} + i\vec{b})$. In the case of helix $k = \tau\tan\vartheta$ it gives

$$C\Omega_3\tau - A\tau^2 = \Delta MH\cos^2\vartheta. \tag{29}$$

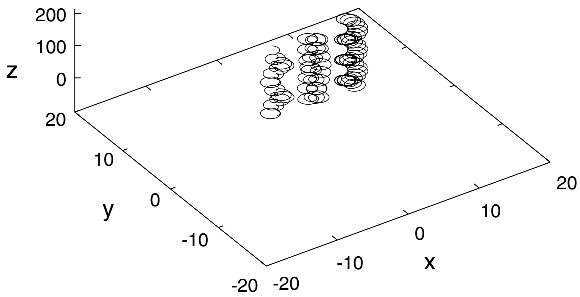

Figure 3. Periodic solutions for twisted ferromagnetic filament. The values of the integral W of configurations from left to right: 0.4442; 0.4385; 0.4226. $Tm = 0.4444$.

The maximum of the left side (29) in dependence on the torsion gives the critical torque for the buckling of the superparamagnetic filament. Since $\Delta M \sim H$, the critical torque for the buckling of the superparamagnetic filament,

$$C\Omega_{3c} = 2\sqrt{A\Delta M H}, \tag{30}$$

grows linearly with the applied field strength. This is different from the behavior of the ferromagnetic filament, for which the critical torque is proportional to the square root of the field strength. Relations $T_z = C\Omega_3$, $T_z = Ak \cdot \sin\vartheta + C\Omega_3 \cos\vartheta$ and (29) give

$$Tm_s = \frac{1}{\cos\vartheta\,(1+\cos\vartheta)^2}, \tag{31}$$

where $Tm_s = \Delta M H A/(C\Omega_3)^2$ is the magnetotwisting number of the superparamagnetic filament. Expressing the rotation angle R (L is the length of the filament),

$$R = L\Omega_3\left(1 + \frac{C}{A}\tan^2\left(\frac{\vartheta}{2}\right)\right), \tag{32}$$

we obtain the following dependence of the rotation angle on the magnetotwisting number ($L_{\mathrm{mag}} = \sqrt{A/\Delta M H}$ is the characteristic magnetic healing length)

$$\frac{RL_{\mathrm{mag}}}{L} = \left(\frac{A}{C} - 1\right) Tm_s^{-1/2} + 2\sqrt{\cos\vartheta}. \tag{33}$$

Expression for the energy of the superparamagnetic helix with the work of the twisting torque $C\Omega_3 R$ taken into account is

$$E = \frac{C\Omega_3^2 L}{2} - C\Omega_3 R + L\left(\frac{A|\psi|^2}{2} - \frac{\Delta M H\cos^2\vartheta}{2}\right).$$

Relations (29) and (32) give the following expression:

$$\Delta E = L\Delta M H\left(-\frac{1-\cos\vartheta}{Tm_s(1+\cos\vartheta)} + \frac{1}{2}\cos\vartheta - \frac{1}{2}\cos\vartheta\sqrt{\frac{\cos\vartheta}{Tm_s}}\right), \tag{34}$$

for the difference of the energies of the straight and the helical superparamagnetic filaments. The dependence of $\Delta E/L\Delta M H$ on the twisting angle RL_{mag}/L is shown in figure 4 ($A/C = 3/2$). We see that the energy of the helical superparamagnetic filament is less than that of the straight filament for all twisting angles. This behavior is different from the behavior of the ferromagnetic filament.

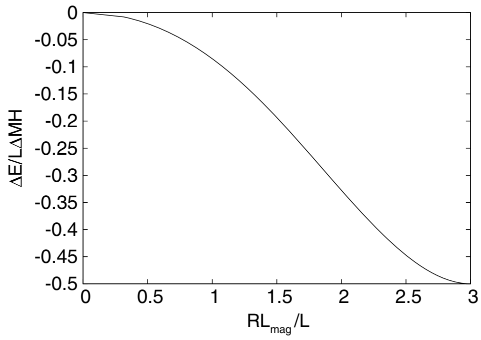

Figure 4. Energy of superparamagnetic helice.

5. Buckling of a filament with finite length

Let us consider the buckling of a filament with a finite length. Scaling the curvature with $1/L$ and the length with L, the equation for the complex curvature of the force free filament under the action of the magnetic field reads

$$-\psi_{,ll} + \mathrm{i}\chi\psi_{,l} - \tfrac{1}{2}|\psi|^2\psi + W\psi = 0. \tag{35}$$

The boundary conditions at the unclamped ends are

$$\psi(\pm 1) = 0. \tag{36}$$

Relations

$$-\psi_{,l}(-1) + \mathrm{i}\chi\psi(-1) - Cm\vec{\epsilon}(-1)\cdot\vec{h} = 0 \tag{37}$$

and

$$W = Cm\vec{e}_3(-1)\cdot\vec{h} \tag{38}$$

allow us to determine the constant W and $\vec{e}_3(-1)\cdot\vec{h}$. Here the magnetoelastic number $Cm = MHL^2/A$ and the twist $\chi = C\Omega_3 L/A$ are introduced. Equation (35) contains only intrinsic property of the filament—its complex curvature—and may be integrated without the knowledge of the filament orientation. Then the unknown integration constant W can be found from relations (37) and (38).

The neutral curve of the buckling instability of the twisted ferromagnetic filament in the plane (Cm, χ) may be calculated from the linearized equation (35) at the boundary conditions $\psi(\pm 1) = 0$; $W = Cm$ and can be found in [2, 6]. Neutral odd and even buckling modes are

$$(\psi_e, \psi_o) = \exp(\mathrm{i}\chi l/2)(\cos(\pi(2n+1)l/2), \qquad \sin(\pi n l)). \tag{39}$$

Here we find the solution of the nonlinear problem (35)–(38). Putting $\psi = \exp(\mathrm{i}\chi(l+1)/2)$ $f(l)$ with a real function f, we have

$$2f_{,ll} + U_{,f} = 0, \tag{40}$$

where

$$U(f) = -\left(W - \frac{\chi^2}{4}\right)f^2 + \frac{1}{4}f^4.$$

Its first integral reads

$$f_{,l}^2 + U = E_0, \tag{41}$$

where $E_0 = f_{,l}^2(-1) = f_{,l}^2(1) \geqslant 0$ due to relation (36). Putting f_0 as the extremal curvature at which $f_{,l} = 0$, we have

$$E_0 = -\left(W - \frac{\chi^2}{4}\right) f_0^2 + \frac{1}{4} f_0^4,$$

where $\chi^2 - 4W + f_0^2 > 0$ due to $E_0 > 0$. As a result, the even solution $(f(l) = f(-l))$ of equation (41) satisfying the boundary conditions $\psi(\pm 1) = 0$ reads ($K(k^2)$ is elliptic integral of type I)

$$f = 2k(2n+1)K(k^2)cn((2n+1)K(k^2)l, k^2), \tag{42}$$

where

$$k^2 = \frac{f_0^2}{2f_0^2 + \chi^2 - 4W}.$$

For the first even mode ($n = 0$) we have

$$f = 2kK(k^2)cn(K(k^2)l, k^2). \tag{43}$$

In a similar way, we obtain the solutions for the odd modes. In particular, the solution for the first odd mode reads

$$f = 4kK(k^2)cn(2K(k^2)(l - 1/2), k^2). \tag{44}$$

Since $\vec{\epsilon} = (\vec{e}_1 + i\vec{e}_2)\exp(i\chi(l+1)/2)$, where $\vec{e}_1(-1) = (\cos\beta, 0, -\sin\beta), \vec{e}_2(-1) = (0, 1, 0), \vec{e}_3(-1) = (\sin\beta, 0, \cos\beta)$ and β is the angle the tangent makes with the magnetic field at the end of the filament and then from (37) and (38) it follows that

$$Cm \sin\beta = f_{,l}(-1) \tag{45}$$

and

$$W = Cm\cos\beta. \tag{46}$$

As a result, we obtain relations

$$\begin{aligned} f_0 &= 2kK(k^2), \\ \cos\beta &= \frac{\chi^2 - (4 - 8k^2)K^2(k^2)}{4Cm}, \\ Cm\sin\beta &= 2kK^2(k^2)\sqrt{1-k^2}, \end{aligned} \tag{47}$$

which give in the parametric form the family of solutions of the nonlinear boundary problem (35)–(38) bifurcating at $\chi_c^2 = \pm 4Cm + \pi^2$ $(Cm > 0)$. For the analysis of the character of these bifurcations, it is convenient to rewrite the last two equations of (47) in the form of a single transcendental equation for angle β:

$$K\left(\frac{1}{2}\left(1 + \frac{Tm\cos\beta - 1/4}{Q^{1/2}}\right)\right) = \sqrt{\frac{Cm}{Tm}}Q^{1/4}, \tag{48}$$

where

$$Q = \left(Tm - \tfrac{1}{4}\right)^2 + \tfrac{1}{2}Tm(1 - \cos\beta).$$

The character of the bifurcation for a given Cm may be analyzed by the series expansion at $Tm_{c1} = Cm/(4Cm + \pi^2)$, which gives

$$1 - \cos\beta = \frac{(Tm/Tm_{c1} - 1)}{1 - Tm_{c1}}\left(\frac{1}{4Tm_{c1}} - 1\right). \tag{49}$$

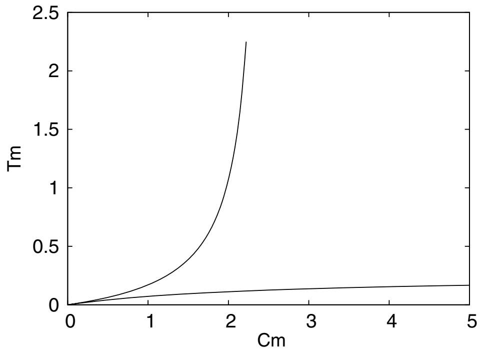

Figure 5. Critical values of the magnetotwisting number as a function of the magnetoelastic number. Tm_{c1}—lower curve and Tm_{c2}—upper curve.

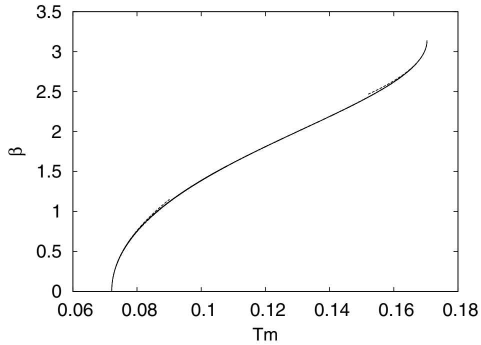

Figure 6. Bifurcation curve of twisted ferromagnetic filament. Dashed lines—asymptotic relations equation (49); (50). $Cm = 1$.

Since $Tm_{c1} < 1/4$, a nontrivial bifurcating solution exists if $Tm > Tm_{c1}$, which corresponds to the region, where the straight filament with the magnetization along the external field is stable. Thus the bifurcation is subcritical.

It is shown in [4] that a straight ferromagnetic filament with the magnetization opposite to the field is unstable with respect to the even U-like mode if $Cm > \pi^2/4$. The critical magnetotwisting number for this case is given by $Tm_{c2} = Cm/(\pi^2 - 4Cm)$. The dependences of Tm_{c1} and Tm_{c2} on Cm are shown in figure 5. For the solution bifurcating at Tm_{c2} we have the series expansion

$$1 + \cos\beta = -\frac{(Tm/Tm_{c2} - 1)}{1 + Tm_{c2}}\left(1 + \frac{1}{4Tm_{c2}}\right). \tag{50}$$

The angle β for the solution bifurcating at Tm_{c1} and Tm_{c2} together with the asymptotic relations (49) and (50) is shown in figure 6 for the characteristic case $Cm = 1$. There is a qualitative difference between the behavior of the filament for $Cm < \pi^2/4$ and for $Cm > \pi^2/4$. In the first case, the solution bifurcating at $(\beta = 0, Tm = Tm_{c1})$ joins the solution bifurcating at $(\beta = \pi, Tm = Tm_{c2})$. In the second case, β reaches the maximum at some intermediate value of the magnetotwisting number. We illustrate this by the plot in the (χ, Cm) plane of the curves corresponding to constant values of angle β (figure 7). With the increase of the angle β, the constant angle curves approaches the dashed line in the lower

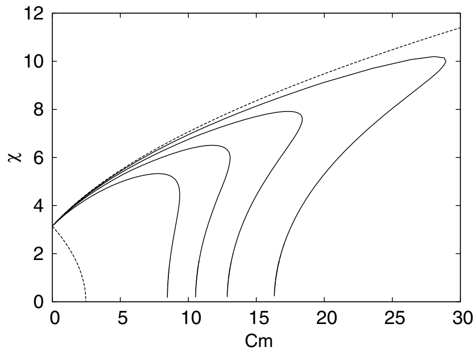

Figure 7. Constant angle curves. From left to right $\cos\beta = 0.9; 0.95; 0.975; 0.99$. Dashed lines—neutral curves $\chi = \sqrt{\pi^2 + 4Cm}$; $\chi = \sqrt{\pi^2 - 4Cm}$.

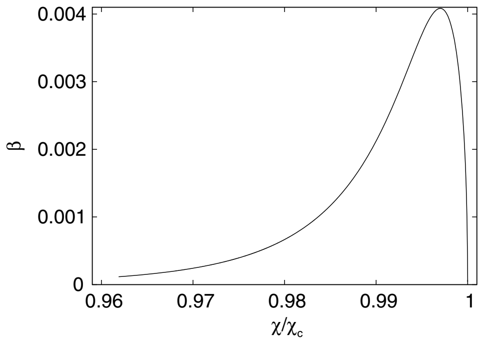

Figure 8. Bifurcation curve of twisted ferromagnetic filament. $Cm = 1000$.

left corner of figure 7. There are no solutions below this curve. For large Cm, the angle β is small and is given by the asymptotic expression $\beta \sim 8\exp(-\sqrt{Cm})$, which follows from equation (48). The characteristic dependence of angle β on the twist for this case of a large Cm equal to 1000 is shown in figure 8.

The shapes of the filaments corresponding to the solutions for the complex curvature can be found by integrating equations (8)–(10) at the boundary conditions:

$$\begin{aligned}
&\vec{e}_3(-1) = (\sin\beta, 0, \cos\beta),\\
&\mathrm{Re}(\vec{\varepsilon}(-1)) = (\cos\beta, 0, -\sin\beta),\\
&\mathrm{Im}(\vec{\varepsilon}(-1)) = (0, 1, 0),\\
&\vec{r}(-1) = (0, 0, 0).
\end{aligned} \tag{51}$$

The evolution of the shapes along the constant angle curves is shown in figure 9 ($\cos\beta = 0.999$) and in figure 10 ($\cos\beta = 1/\sqrt{2}$) for the increasing values of the magnetotwisting number. We see that the shape evolves from the loop perpendicular to the field (figure 9) to the loop in the plane of the field with the magnetization opposite to the direction of the magnetic field. The shapes at $Tm \to \infty$ correspond to the magnetic elastica considered in [26] and, as shown there, are unstable. For large Cm and Tm close to $1/4$, coiled shapes are formed, as is illustrated by figure 11, where the buckled ferromagnetic filament at $Cm = 1000$ and $\chi/\chi_c = 0.9975$ is shown.

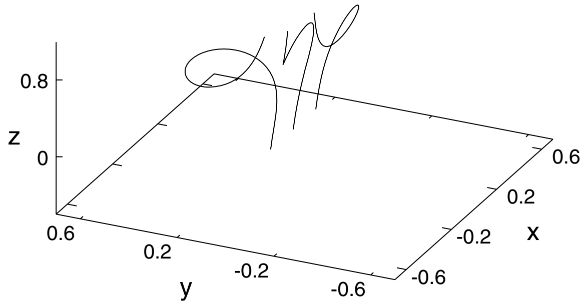

Figure 9. Shapes of buckled filaments. From left to right: $Tm = 1, 10, 100$ ($\cos\beta = 0.999$).

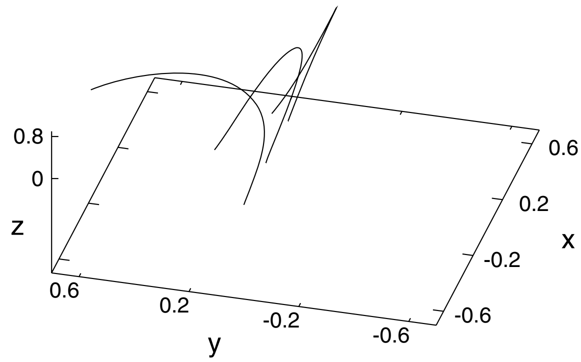

Figure 10. Shapes of buckled filaments. From left to right: $Tm = 0.5, 10, 100$ ($\cos\beta = 1/\sqrt{2}$).

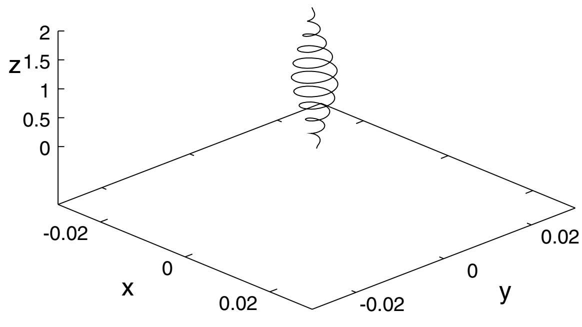

Figure 11. Buckled twisted ferromagnetic filament. $Cm = 1000$; $\chi/\chi_c = 0.9975$.

Since at $k \to 1 K(k^2) \simeq \sqrt{4Cm - \chi^2}/2$ and $cn(x, k^2) \to ch^{-1}(x)$, the solution of the boundary problem for large $\sqrt{4Cm - \chi^2}$ approaches the localized solution

$$f = \chi\sqrt{4Tm - 1}\frac{1}{ch(\chi\sqrt{4Tm - 1}l/2)}. \qquad (52)$$

Taking into account that in the limit of $L \to \infty$ the characteristic distance is determined by L_*, this solution coincides with (17).

The solutions corresponding to the real function f are the only possible solutions for the curvature of the ferromagnetic filament with the free and unclamped ends. Indeed, if $f = A\exp(\mathrm{i}\varphi)$ ($\psi = f\exp(\mathrm{i}\chi l/2)$), where $\varphi_{,l} \neq 0$, $A = A(l) \geqslant 0$, then equation (35) is equivalent to two equations for A and φ:

$$\varphi_{,ll}A + 2\varphi_{,l}A_{,l} = 0 \tag{53}$$

and

$$-A_{,ll} + \varphi_{,l}^2 A + \left(W - \frac{\chi^2}{4}\right)A - \frac{1}{2}A^3 = 0. \tag{54}$$

Equation (54) by the integration of equation (53) according to $\varphi_{,l} = c/A^2$ may be put into the form

$$2A_{,ll} + U_{,A} = 0, \tag{55}$$

where

$$U(A) = -\left(W - \frac{\chi^2}{4}\right)A^2 + \frac{A^4}{4} + \frac{c^2}{A^2}.$$

Its first integral reads

$$A_{,l}^2 + U = E_0. \tag{56}$$

$U(A)$ has only one minimum for positive A and tends to $+\infty$ at $A \to 0$ and $A \to +\infty$. This means that the solution with $c \neq 0$ corresponds to the curvature in the interval $[A_1; A_2](U(A_1) = U(A_2) = E_0(0 < A_1 < A_2))$ and cannot satisfy the boundary condition at the unclamped ends. Thus, the family of solutions found above is the only possible for the filament with free and unclamped ends.

The solution of the equations of equilibrium of the free twisted ferromagnetic rod under the action of the homogeneous field allows one to construct the solutions of other problems, for example, to obtain the solution for the equilibrium of the rod with the hinged ends.

The condition of the equilibrium $\vec{F}_{,l} = 0$ gives $\vec{F} = \vec{F}_0$. This, using the identity $\vec{F}_0 = \mathrm{Re}(\varepsilon^* \vec{F}_0 \cdot \vec{\varepsilon}) + \vec{e}_3 \vec{F}_0 \cdot \vec{e}_3$ and the expression for the stress equation (2), gives

$$-A\psi_{,l} + \mathrm{i}C\Omega_3\psi - MH_0\vec{h}_0 \cdot \vec{\epsilon} = \vec{F}_0 \cdot \vec{\varepsilon}, \tag{57}$$

$$\tfrac{1}{2}A|\psi|^2 + \Lambda = \vec{F}_0 \cdot \vec{e}_3. \tag{58}$$

Equations (57), (58) and the integral $W = -\Lambda + MH_0\vec{h}_0 \cdot \vec{e}_3 = \mathrm{const}$ show that the problem of the equilibrium shape of the filament under the action of the magnetic field $\vec{H}_0$ and a constant force $\vec{F}_0$ is equivalent to the problem of the equilibrium shape of a free rod in the effective magnetic field $\vec{H}$: $M\vec{H} = \vec{F}_0 + M\vec{H}_0$.

The equilibrium shapes of the free twisted filaments are parametrized by the dependence of the tangent angle at the end of the filament on the dimensionless parameters Cm and Tm. In the case of the rod with the hinged ends, the values of these parameters are unknown and should be found from the additional condition. This condition reads $\vec{F}_0 \cdot (\vec{r}(L) - \vec{r}(-L)) = 0$ and expresses the condition on the force applied at the hinge to be perpendicular to the radius–vector connecting the ends of the filament. The condition of the hinge gives the necessary equation for the force and thus the equilibrium solution for the force free ferromagnetic filament allows one to construct solutions for the boundary conditions with the hinged ends. The application of this principle for the analysis of the equilibrium shapes of twisted rods with the hinged ends will be considered in the forthcoming publication.

6. Conclusion

To conclude, we have found the family of buckled static shapes of the free ferromagnetic filaments with unclamped ends, which bifurcates at values of parameters corresponding to the threshold of linear instability of the straight configuration. The transition to the buckled shape is subcritical in the case of magnetization along the field and supercritical when the magnetization of the straight filament is opposite to the external field. At large values of the magnetoelastic number, the shape of the ferromagnetic filament with the decrease of the twist evolves from the shape with a loop oriented perpendicularly to the field to shape with a loop in the plane of the field. Magnetization in the central part of the loop is opposite to the field. Solutions corresponding to the helical shapes of the ferromagnetic and superparamagnetic filaments are found, which show that the helical shape of a twisted superparamagnetic filament is energetically more advantageous in comparison with the straight filament for all values of the twist. It is remarked that similarly to an extensible string with the electric current the ferromagnetic string at the equilibrium has helical shape under the action of external field. Solutions of equations of equilibrium of the free twisted ferromagnetic filaments under the action of the magnetic field allow us to construct the solutions for other problems, for example, calculate equilibrium shapes of twisted filaments with hinged ends.

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